

Ques :- State and Prove Existence theorem and uniform limit

Solⁿ :-

Existence Theorem :- Let $F(x, y)$ be a real valued function of variables x, y

where (x, y) varies over $S \times T \subseteq \mathbb{R}^2$ and

let $a \in \bar{S}, b \in \bar{T}$

If $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} F(x, y) = l$ exists and if any inner

limit $\lim_{y \rightarrow b} F(x, y)$

($\lim_{x \rightarrow a} F(x, y)$ exists then corresponding repeated limit

$l_2, (l_{1,2})$ exists and is equal to l .

proof :- Let $\lim_{y \rightarrow b} F(x, y) = l_2(x), x \in S$

Since also $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} F(x, y) = l$ (by hypothesis) for any $\epsilon > 0 \exists \delta > 0 \exists \eta > 0$

such that

$$|F(x, y) - l| < \frac{\epsilon}{2} \text{ and } |F(x, y) - l_2(x)| < \frac{\epsilon}{2}$$

for $x \in V(a) \text{ \& } y \in V(b)$ — (1)

Now $|l_2(x) - l| \leq |l_2(x) - F(x, y)| + |F(x, y) - l|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ — (11)}$$

provided $x \in V(a) \text{ \& } y \in V(b)$

But since the L.H.S expression is independent of y result
 (ii) means that for any $\varepsilon > 0 \exists \delta > 0$ such that

$$|I_2(x) - 1| < \varepsilon, x \in V(a) \therefore I_2 = \lim_{x \rightarrow a} I_2(x) = 1$$

Thus completes the proof.

Def uniform limit:-

Suppose $\lim_{x \rightarrow a} F(x, y) = g(y), y \in T$

This means that for each fixed $y \in T$ corresponding to any $\varepsilon > 0, \exists \delta(y)$ such that

$$|F(x, y) - g(y)| < \varepsilon, \text{ provided } |x - a| < \delta(y)$$

Symbolically we express this as follows:-

$$(\varepsilon > 0) (y) (\exists \delta) [|F(x, y) - g(y)| < \varepsilon \text{ provided } |x - a| < \delta]$$

Here as is obvious δ depends not only upon ε but upon the y as well.

However if in certain cases δ depends upon ε only (but not upon y)

$$\text{i.e. } \delta = \delta(\varepsilon) \text{ provided } |x - a| < \delta$$

then we say that

$$\lim_{x \rightarrow a} f(x, y) = g(y)$$

uniformly over T .